ON THE DENSE PACKING OF SPHERES

BY

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1. Definitions and statement of the theorem. Let

$$(1) \qquad\qquad S_1, \cdots, S_n$$

be a set of spheres with radii of lengths

$$(1.1) \qquad\qquad r_1 \leqq r_2 \leqq \cdots \leqq r_n$$

respectively. Let S be a right circular cylindrical surface with radius of length r, such that $r_n \leqq r$. A sphere S_i will be said to be *inside* S if its interior is completely in the interior of S. Let the spheres (1) be assigned positions inside S, no two having interior points in common. We say that the spheres (1) have been *packed* into S, and shall be referred to as a *packing*. All packings considered in this paper are subject to the restriction:

(A) *Let* $[pq] \equiv$ *distance between the centers of S_p and S_q. Let*

$$(2) \qquad\qquad S_i,\ S_j,\ S_k \qquad\qquad (\in (1))$$

be in S with (a) $[ij] = r_i + r_j$, (b) $[jk] = r_j + r_k$, (c) $[ik] \geqq r_i + r_k$. *Then the spheres* (1) *are assumed to be so large that S_i, S_j, S_k cannot be continuously moved in S, keeping* (a) *and* (b) *true, into a position where* (c) *is false* (cf. §4).

COROLLARY 1. *If x, y and z are the respective projections of the centers of the spheres* (2) *on to the axis of S, then y "separates" x and z.*

COROLLARY 2. *Let r_i and r_j be any elements of* (1.1), *then $r_i + r_j > r$.*

COROLLARY 3. *For any element of* (1) $r_i \geqq r/4$.

Let the spheres (1) be packed into S. Let x and y denote the projections of the centers of the spheres S_i and S_j of the packing on to l, the axis of S. We say that S_i and S_j are *adjacent* if the projection on to l of no other center of the spheres in the packing separates x and y. Let D denote the smallest right circular frustrum of S and its interior, which encloses the packing. The *density* of the packing is here defined as the volume ratio of the spheres to D.

A packing will be called *incompressible* if any two adjacent spheres are (1) tangent to each other and (2) each tangent to S at diametrically opposite elements of S.

COROLLARY. *The centers of the spheres forming an incompressible packing all lie in one axial plane of S.*

14

Let a packing P of the spheres (1) into S be given. Let l be oriented. Then P will be denoted by an ordered array of marks $S_{i_1}, \cdots, S_{i_n}$ written from left to right so that (1) adjacent marks denote adjacent spheres, and (2) if x and y are the projections of the centers of S_{i_j} and S_{i_k}, where $j < k$, then the orientation of the directed line segment $\overrightarrow{xy}$ is positive. It is obvious that the density of a packing $S_{i_1}, \cdots, S_{i_n}$ and its reverse $S_{i_n}, \cdots, S_{i_1}$ are the same.

We prove the following theorem.

THEOREM. *Consider the set of all possible packings of the spheres* (1) *into S, subject to the restriction* (A).

(1) *The density is greatest for the incompressible packing*

$$(3) \qquad \{ \cdots S_{2i} \cdots S_6 S_4 S_2 S_1 S_3 S_5 S_7 \cdots S_{2i+1} \cdots \}.$$

(2) *The least dense incompressible packing is*

$$(4) \qquad \{ S_1 S_n S_3 S_{n-2} S_5 S_{n-4} \cdots S_{n-5} S_6 S_{n-3} S_4 S_{n-1} S_2 \}.$$

It is clear that if some of the spheres (1) have the same radius, then there are other packings whose densities are equal to those of (3) or (4). If there are no spheres in (1) with equal radii, then it will be clear from the proof that no other packing can have a density equal to that of (3) or (4) (except their reverses).

We note that without the restriction (A) we obtain one of the unsolved packing problems: How can n spheres be packed into a container in the densest fashion? The problem remains unsolved even for the case where the container is a cube or cylinder, and the spheres are equal.

2. **Some inequalities.** We prove the following lemma.

LEMMA 1. *Let a, b, c, d, r be positive constants such that $a+b$, $b+c$, $c+d$, $d+a$ are each greater than r. If $a < c$ and $b < d$, then*

$$(5) \qquad (a + b - r)^{1/2} + (c + d - r)^{1/2} < (b + c - r)^{1/2} + (a + d - r)^{1/2}.$$

REMARK. It is obvious that if $a = c$, or $b = d$, then the inequality (5) becomes an equality.

Proof. Let K and L be real constants with $K < L$. Let $x > L$. Then, $(x - K)^{1/2} > (x - L)^{1/2}$. Let $f(x) \equiv (x - K)^{1/2} - (x - L)^{1/2}$. Then $f'(x) < 0$. Thus $f(x)$ is a monotonically decreasing function (in the strong sense) in $L < x < \infty$. Thus if $x_2 > x_1 > L$

$$(6) \qquad (x_2 - K)^{1/2} - (x_2 - L)^{1/2} < (x_1 - K)^{1/2} - (x_1 - L)^{1/2}.$$

Now let $K = r - c$ and $L = r - a$. Since $a < c$, then $K < L$. Let $x_2 = d$ and $x_1 = b$. Since $b < d$, then $x_1 < x_2$. Substituting into (6) we obtain

(7) $(d - (r - c))^{1/2} - (d - (r - a))^{1/2} < (b - (r - c))^{1/2} - (b - (r - a))^{1/2}$.

Transposing the second term of each side, we obtain (5).

LEMMA 2. *Let a, b, c, r be positive constants such that $a+b$ and $a+c$ are each greater than r. If $b < c \leqq r$ and $a \leqq r$ then*

(8) $$c + 2(r(a + b - r))^{1/2} < b + 2(r(a + c - r))^{1/2}.$$

REMARK. If $b = c$, then (8) becomes an equality.

Proof. Since $c - b > 0$ and b and c are positive, $c + b > 2(bc)^{1/2}$. Thus $2c + b > c + 2(bc)^{1/2}$, and $2c - 2(bc)^{1/2} > c - b$. Therefore

(9) $\quad c^{1/2} - b^{1/2} = (2c - 2(bc)^{1/2})/2c^{1/2} > (c - b)/2c^{1/2} \geqq (c - b)/2r^{1/2}$.

If $a = r$ then (9) implies (8). Suppose $a < r$. In the proof of Lemma 1 we saw that if $K < L$, $f(x) \equiv (x - K)^{1/2} - (x - L)^{1/2}$ is a monotonically decreasing function (in the strong sense) for $L < x < \infty$. In (6) let $K = 0$, $L = r - a$, $x_1 = b$, $x_2 = c$. Transposing, we obtain

(10) $$(a + c - r)^{1/2} - (a + b - r)^{1/2} > c^{1/2} - b^{1/2}.$$

Combining (9) and (10) we obtain

(11) $$(a + c - r)^{1/2} - (a + b - r)^{1/2} > (c - b)/2r^{1/2}.$$

Multiplying both sides by $2r^{1/2}$ and transposing, we obtain (8).

LEMMA 3. *Let K be a positive constant. If*

(12) $$0 < x \leqq K, \qquad 0 < y \leqq K, \qquad 0 < z \leqq K,$$
$$x + y > K, \qquad x + z > K,$$

then

(12.1) $\quad f(x, y, z) \equiv (K(y + x - K))^{1/2} + (K(x + z - K))^{1/2} - (yz)^{1/2}$

is a monotonically increasing function, that is, if (x_1, y_1, z_1) and (x_2, y_2, z_2) are in the region (12) and if $x_1 \leqq x_2$, $y_1 \leqq y_2$, $z_1 \leqq z_2$, then

(13) $$f(x_1, y_1, z_1) \leqq f(x_2, y_2, z_2).$$

Proof. We note that if

(14) $$x_1 \leqq x \leqq x_2, \quad y_1 \leqq y \leqq y_2, \quad \text{and} \quad z_1 \leqq z \leqq z_2,$$

then (x, y, z) is also in the region (12).

By the theorem of the mean for functions of three variables

$$f(x_2, y_2, z_2) = f(x_1, y_1, z_1) + h \cdot f_x(x_1 + \theta h, y_1 + \theta k, z_1 + \theta l)$$
$$+ k \cdot f_y(x_1 + \theta h, y_1 + \theta k, z_1 + \theta l)$$
$$+ l \cdot f_z(x_1 + \theta h, y_1 + \theta k, z_1 + \theta l)$$

where $0 < \theta < 1$ and $h = x_2 - x_1$, $k = y_2 - y_1$, $l = z_2 - z_1$. But

$$f_x(x_1 + \theta h, \, y_1 + \theta k, \, z_1 + \theta l) \equiv (K^{1/2}/2) \cdot ((x_1 + y_1 + \theta(h + k) - K)^{-1/2}$$
$$+ (x_1 + y_1 + \theta(h + l) - K)^{-1/2}) > 0.$$

Also,

$$f_y(x_1 + \theta h, \, y_1 + \theta k, \, z_1 + \theta l) \equiv 2^{-1}(K^{1/2}(x_1 + y_1 + \theta(h + k) - K)^{-1/2}$$
$$- (z_1 + \theta l)^{1/2} \cdot (y_1 + \theta k)^{-1/2}) \geqq 0$$

since $x_1 + \theta h < K$ and $z_1 + \theta l < K$. Similarly,

$$f_z(x_1 + \theta h, \, y_1 + \theta k, \, z_1 + \theta l) \geqq 0.$$

Thus (13) is true.

3. **A minimal property of incompressible packings.** Let two spheres A and B with centers a and b, and radii of lengths α and β respectively, be packed into a cylinder S. Let l be the axis of S, and r the length of the radius. Suppose also that $\alpha + \beta > r$. Let x and y be the projections of a and b respectively upon l. We prove the following lemma.

LEMMA 4. *The distance between x and y is a minimum if and only if the packing A, B is incompressible.*

Proof. If A and B are not mutually tangent, then it is obvious that by translating each parallel to l and toward each other, that $d(x, y)$ (the distance between x and y) is decreased. Thus let us suppose that A and B are mutually tangent. Suppose one of the spheres, say B, is not tangent to S. We show how A and B can be rearranged to make $d(x, y)$ smaller, and B tangent to S. Let the ray emanating from a, which is parallel to $\overrightarrow{xy}$ and similarly oriented, intersect the sphere D with a as center and radius $(\alpha + \beta)$ in the point p. If $p \equiv b$, let G be any great circle on D through p. If $p \not\equiv b$, let G be the great circle on D through b and p. Let S' be the right circular cylinder with l as axis and radius $(r - \beta)$. Since $\alpha + \beta > r$, D intersects S'. Starting from p, let us traverse G going toward b along the minor arc pb. Let w_1 be the first intersection point with S', and w its projection upon l. Since B is not tangent to S, w falls between x and y. Thus, if B is moved so that its center falls on w_1, then the distance between the projections of the centers of A and B upon l is reduced, that is, it becomes $d(x, y) - d(w, y)$.

Let us now suppose that A and B are each tangent to S. Let S be taken with l on the x-axis, with a on the positive half of the y-axis at the point $(0, r - \alpha, 0)$, and b at the point (x, y, z) with $x > 0$. Then the equation of D is

(15) $$x^2 + (y - (r - \alpha))^2 + z^2 = (\alpha + \beta)^2$$

and the equation of S' is

(16) $$y^2 + z^2 = (r - \beta)^2.$$

Eliminating y between (15) and (16), we see that $(dx/dz) = 0$ if and only $z = 0$, with the minimum value of x being

$$(17) \qquad 2(r(\alpha + \beta - r))^{1/2}.$$

Thus, that point of the curve defined by (15) and (16) which is closest to the yz-plane is

$$(18) \qquad (2(r(\alpha + \beta - r))^{1/2}, \beta - r, 0).$$

This means that if the sphere B is put so that its center falls on the point (18), A and B will be mutually tangent, and each tangent to S at diametrically opposite elements. Thus when the packing is incompressible, the projection on to l of the line segment joining a to b is a minimum, and is given by (17).

If by translations parallel to, and rotations about, the axis of S, one packing can be carried into (that is, becomes identical to) another, then we say that the packings are *equivalent*. It is obvious that the densities of two equivalent packings are identical.

Two packings of the spheres (1) into S

$$S_{i_1}, \cdots, S_{i_n} \quad \text{and} \quad S_{j_1}, \cdots, S_{j_n}$$

will be said to be *similarly ordered* if (1) $S_{i_k} \equiv S_{j_k}$ and (2) if the projections of the centers of $S_{i_1}, S_{i_n}, S_{j_1}, S_{j_n}$ on to the axis l, are a, b, c, d respectively, then the orientation of $\overrightarrow{ab}$ is the same as that of $\overrightarrow{cd}$.

"Similar ordering" is an equivalence relation. Let $\{S\}$ denote the set of all packings of the spheres (1) into S. Then $\{S\}$ can be subdivided into maximal sets of similarly ordered packings:

$$(19) \qquad \{S\} \equiv \{S\}_1 + \cdots + \{S\}_{nl}$$

where any two elements of $\{S\}_i$ have similar orderings, and any two elements from distinct sets do not. It is clear that in any $\{S\}_i$ there is one and only one class of equivalent incompressible packings.

COROLLARY. *Let $\{S\}_i$ be any one of the maximal sets of similarly ordered packings of (19). Then the density of any element of $\{S\}_i$ which is not incompressible is less than that of the incompressible packing (that is, any element of the class of equivalent incompressible packings).*

Proof. Let

$$(20) \qquad S_{i_1}, \cdots, S_{i_n}$$

be an element of $\{S\}_i$ which is not incompressible. Let $p_1, \cdots, p_n$ denote the projections of the respective centers of (20) on to the axis of S. Then by Lemma 4, if the subpacking $S_{i_j} S_{i_{j+1}}$ is not incompressible

$$d(p_j, p_{j+1}) > 2(r(r_{i_j} + r_{i_{j+1}} - r))^{1/2}.$$

Since (20) is not incompressible, there must be at least one pair of adjacent spheres which do not form an incompressible subpacking. Thus

$$r_1 + \left(\sum_{j=1}^{n-1} d(p_j, p_{j+1}) \right) + r_n > r_1 + \left(\sum_{j=1}^{n-1} 2(r(r_{i_j} + r_{i_{j+1}} - r))^{1/2} \right) + r_n,$$

from which the corollary follows.

We note that if a packing of the spheres (1) into S is densest, then it must be incompressible.

4. The assumption expressed analytically. We prove that *the inequality*

$$(21) \qquad (r(r_1 + r_2 - r))^{1/2} + (r(r_1 + r_3 - r))^{1/2} \geqq (r_2 r_3)^{1/2}$$

is equivalent to the assumption (A).

Proof. The assumption states that for *any* relative positions of the packing (2) satisfying the hypothesis of (A), the conclusion holds. Then certainly it holds for those arrangements where the centers of S_i and S_k are closest. Suppose that the spheres (2) form an incompressible packing in the given order (that is, the projection of the center of S_j on to l falls between the projections of the other two centers upon l). Then if the section of this packing with the plane on which the centers lie is considered, it is seen that the assumption implies that

$$(22) \qquad (r(r_i + r_j - r))^{1/2} + (r(r_j + r_k - r))^{1/2} \geqq (r_i r_k)^{1/2}$$

for distinct i, j and k. Conversely, if (22) is true, and if the packing (2) is incompressible, then S_i and S_k cannot have any interior point in common. But for any other similarly ordered packing of the spheres (2), the centers of the end elements cannot be closer than for the incompressible packing. Thus (22) implies the assumption (A).

We now note that if $f(x, y, z)$ is the function defined by (12.1), and if $r_b \leqq \min (r_a, r_c)$, then

$$(23) \qquad f(r_b, r_a, r_c) \leqq f(r_a, r_b, r_c)$$

since $(r(r_b + r_c - r))^{1/2} \leqq (r(r_a + r_c - r))^{1/2}$ and $(r_a r_c)^{1/2} \geqq (r_b r_c)^{1/2}$. Also,

$$(24) \qquad f(r_b, r_a, r_c) \leqq f(r_c, r_a, r_b).$$

Thus, since $r_1 \leqq r_2 \leqq r_3$,

$$(25) \qquad f(r_1, r_2, r_3) \leqq \min (f(r_2, r_1, r_3), f(r_3, r_2, r_1)).$$

Now, let r_i, r_j and r_k be distinct elements of (1.1). By Lemma 3, and bearing in mind that $f(x, y, z) = f(x, z, y)$, we have

$$f(r_i, r_j, r_k) \geqq f(r_1, r_2, r_3) \qquad \text{if } i < j < k \text{ or } i < k < j,$$

$$f(r_i, r_j, r_k) \geqq f(r_2, r_3, r_1) \qquad \text{if } i < i < k \text{ or } k < i < j,$$

and

$$f(r_i, r_j, r_k) \geqq f(r_3, r_1, r_2) \qquad \text{if} \quad k < j < i \quad \text{or} \quad j < k < i.$$

Thus by (25)

(26) $$f(r_i, r_j, r_k) \geqq f(r_1, r_2, r_3).$$

But if (22) is true for all distinct i, j, k, then certainly (21) is true. Conversely if (21) is true, by (26), (22) is true for all $i, j,$ and k. Thus the assumption (A) is equivalent to (21).

We note that (21) tells us explicitly how large the spheres must be taken.

5. **Proof of part (1) of the theorem.** By the corollary of §3, it becomes necessary to consider only the incompressible packings. Henceforth the term "packing" will mean "incompressible packing." Let P_1 be any packing of the spheres (1) into S. We shall prove part (1) by constructing an ordered set of packings

(27) $$P_1, \cdots, P_n$$

such that

(a) P_i is obtained from P_{i-1} by a rearrangement of the spheres of P_{i-1} $(i = 2, \cdots, n)$,

(b) P_n is the packing (3), and

(c) if d_i is the density of P_i $(i = 1, \cdots, n)$, then

$$d_1 \leqq d_2 \leqq \cdots \leqq d_n.$$

The packing P_ν will be denoted by

(28) $$S_1^\nu, \cdots, S_n^\nu.$$

The symbol

(28.1) $$(\nu; i_1, \cdots, i_n)$$

will denote the packing

(28.2) $$S_{i_1}^\nu, S_{i_2}^\nu, \cdots, S_{i_n}^\nu.$$

$r_{i_j}^\nu$ will denote the length of the radius of $S_{i_j}^\nu$. Notice that the superscript of r will be used as an identification mark, and should not be taken as its power.

Let P_1 be a packing $S_{i_1}, \cdots, S_{i_n}$ of the spheres (1) which we denote by

(29) $$S_1^1, \cdots, S_n^1,$$

that is, $S_j^1 \equiv S_{i_j}$, for the sake of a uniform notation. Let $S_{g_1}^1 \equiv S_1$ and $S_{k_1}^1 \equiv S_2$ (S_i without the superscript denotes the ith sphere from the left in (1)). If $k_1 > g_1$, then we can reverse the packing. If $k_1 = g_1 - 1$ then we can take $P_2 \equiv P_1$.

(1) $1 < k_1 \leqq g_1 - 2$. We show that the density of

(30) $(1; 1, 2, \cdots, k_1 - 1, g_1 - 1, g_1 - 2, \cdots, k_1 + 1, k_1, g_1, g_1 + 1, \cdots, n)$

is greater than or equal to that of (29). For, all the r_i $(i = 1, \cdots, n)$ are positive, and $r_i + r_j > r$. But $r_{g_1}^1 \equiv r_1$ and $r_{k_1}^1 \equiv r_2$. Therefore by (1.1) $r_{g_1}^1 \leqq r_{k_1-1}^1$ and $r_{k_1}^1 \leqq r_{g_1-1}^1$. By Lemma 1

(31)
$$(r_{g_1}^1 + r_{k_1}^1 - r)^{1/2} + (r_{k_1-1}^1 + r_{g_1-1}^1 - r)^{1/2}$$
$$\leqq (r_{k_1-1}^1 + r_{k_1}^1 - r)^{1/2} + (r_{g_1-1}^1 + r_{g_1}^1 - r)^{1/2}.$$

Thus,   .

(32)
$$r_1^1 + \left(\sum_{j=1}^{n-1} 2(r(r_j^1 + r_{j+1}^1 - r))^{1/2} \right) + r_n^1$$
$$\geqq r_1^1 + \left(\sum_{j=1}^{k_1-2} 2(r(r_j^1 + r_{j+1}^1 - r))^{1/2} \right)$$
$$+ 2(r(r_{k_1-1}^1 + r_{g_1-1}^1 - r))^{1/2} + \left(\sum_{j=k_1}^{g_1-2} 2(r(r_j^1 + r_{j+1}^1 - r))^{1/2} \right)$$
$$+ 2(r(r_{k_1}^1 + r_{g_1}^1 - r))^{1/2} + \left(\sum_{j=g_1}^{n-1} 2(r(r_j^1 + r_{j+1}^1 - r))^{1/2} \right) + r_n^1.$$

Therefore the density of (30) is greater then or equal to that of (29).

(2) $k_1 = 1$. The density of

(33) $(1; g_1 - 1, g_1 - 2, \cdots, 1, g_1, g_1 + 1, \cdots, n)$

is greater than or equal to that of (29). $r_{g_1}^1 = r_1$ and $r_1^1 = r_2$. Therefore $r_1^1 \leqq r_{g_1-1}^1$. Since the other conditions of Lemma 2 are also satisfied,

(34) $r_{g_1-1}^1 + 2(r(r_1^1 + r_{g_1}^1 - r))^{1/2} \leqq r_1^1 + 2(r(r_{g_1-1}^1 + r_{g_1}^1 - r))^{1/2}.$

Thus the left side of (32) is greater than or equal to

(35)
$$r_{g_1-1}^1 + \left(\sum_{j=1}^{g_1-2} 2(r(r_j^1 + r_{j+1}^1 - r))^{1/2} \right) + 2(r(r_1^1 + r_{g_1}^1 - r))^{1/2}$$
$$+ \left(\sum_{j=g_1}^{n-1} 2(r(r_j^1 + r_{j+1}^1 - r))^{1/2} \right) + r_n^1.$$

Therefore the density of (33) is greater than or equal to that of (29).

Now let P_2 be the packing (30) or (33) according as k_1 is greater than or equal to 1, and denote it by

(36) $S_1^2, \cdots, S_n^2,$

that is, S_j^2 is identical to the jth element from the left in either (30) or (33). Let $S_{g_2}^2 \equiv S_1$, $S_{g_2-1}^2 \equiv S_2$ and $S_{k_2}^2 \equiv S_3$. If $k = g_2 + 1$, then we can take $P_3 \equiv P_2$.

(1) $g_2+1 < k_2 < n$. Then the density of

(37) $(2; 1, 2, \cdots, g_2, k_2, k_2 - 1, \cdots, g_2 + 2, g_2 + 1, k_2 + 1, k_2 + 2, \cdots, n)$

is greater than or equal to that of P_2. For by Lemma 1

$$(r_{g_2}^2 + r_{k_2}^2 - r)^{1/2} + (r_{g_2+1}^2 + r_{k_2+1}^2 - r)^{1/2}$$
$$\leqq (r_{g_2}^2 + r_{g_2+1}^2 - r)^{1/2} + (r_{k_2}^2 + r_{k_2+1}^2 - r)^{1/2}$$

since $r_{g_2}^2 \leqq r_{k_2+1}^2$ and $r_{k_2}^2 \leqq r_{g_2+1}^2$.

(2) $k_2 = n$. Then the density of

(38) $(2; 1, 2, \cdots, g_2, n, n - 1, \cdots, g_2 + 1)$

is greater than or equal to that of P_2. For by Lemma 2

$$r_{g_2+1}^2 + 2(r(r_{g_2}^2 + r_n^2 - r))^{1/2} \leqq r_n^2 + 2(r(r_{g_2}^2 + r_{g_2+1}^2 - r))^{1/2}$$

since $r_n^2 \leqq r_{g_2+1}^2$.

(3) $1 \leqq k_2 < g_2 - 2$. Then by Lemma 1 or Lemma 2, the density of

(39) $(2; 1, 2, \cdots, k_2 - 1, g_2 - 2, g_2 - 3, \cdots, k_2 + 1, k_2, g_2 - 1, g_2, \cdots, n)$

is greater than or equal to that of P_2, since $r_{g_2-1}^2 \leqq r_{k_2-1}^2$ and $r_{k_2}^2 \leqq r_{g_2-2}^2$. By Lemma 1 or Lemma 2, the density of

(40) $(2; 1, 2, \cdots, k_2-1, g_2-2, g_2-3, \cdots, k_2+1, k_2, g_2, g_2-1, g_2+1, \cdots, n)$

is greater than or equal to that of (39), since $r_{g_2}^2 \leqq r_{g_2-1}^2$ and $r_{k_2}^2 \leqq r_{g_2+1}^2$. Finally, the density of the reverse of (40)

(41) $(2; n, n - 1, \cdots, g_2 + 1, g_2 - 1, g_2, k_2, k_2 + 1, \cdots,$
$$g_2 - 3, g_2 - 2, k_2 - 1, k_2 - 2, \cdots, 1)$$

is the same as that of (40).

We consider the general case, that is, we show how to construct $P_{\nu+1}$ from P_ν.

Case I. $\nu = 2m+1$. Let

(42) $P_\nu: \quad S_1^{2m+1}, \cdots, S_n^{2m+1}$

be such that

(43) $S_{g_\nu+i}' \equiv S_{2i+1}$ $(i = 0, 1, \cdots, m)$, and $S_{g_\nu-i}' \equiv S_{2i}$ $(i = 1, 2, \cdots, m)$.

Let $S_{k_\nu}^{2m+1} \equiv S_{2m+2}$. If $k_\nu = g_\nu - m - 1$, then let $P_{\nu+1} \equiv P_\nu$.

(1) Suppose $1 \leqq k_\nu \leqq g_\nu - m - 2$. Then by Lemma 1 or Lemma 2 the density of

$$(44) \quad (2m + 1; 1, 2, \cdots, k_\nu - 1, g_\nu - m - 1, g_\nu - m - 2, \cdots,$$
$$k_\nu + 1, k_\nu, g_\nu - m, g_\nu - m + 1, \cdots, n)$$

is greater than or equal to that of P_ν, since $r'_{k_\nu-1} \geq r'_{0_\nu-m}$ and $r'_{0_\nu-m-1} \geq r'_{k_\nu}$.

(2) Suppose $g_\nu + m + 2 \leq k_\nu \leq n$. Then by Lemma 1 or Lemma 2 the density of

$$(45) \quad (2m + 1; 1, 2, \cdots, g_\nu + m, k_\nu, k_\nu - 1, \cdots,$$
$$g_\nu + m + 2, g_\nu + m + 1, k_\nu + 1, k_\nu + 2, \cdots, n)$$

is greater than or equal to that of P_ν; also the density of

$$(46) \quad (2m + 1; 1, 2, \cdots, g_\nu - m - 1, g_\nu + m, g_\nu + m - 1, \cdots, g_\nu - m + 1,$$
$$g_\nu - m, k_\nu, k_\nu - 1, \cdots, g_\nu + m + 2, g_\nu + m + 1,$$
$$k_\nu + 1, k_\nu + 2, \cdots, n)$$

is greater than or equal to that of (45). Finally the density of the reverse of (46)

$$(47) \quad (2m + 1; n, n - 1, \cdots, k_\nu + 2, k_\nu + 1, g_\nu + m + 1, g_\nu + m + 2, \cdots,$$
$$k_\nu - 1, k_\nu, g_\nu - m, g_\nu - m + 1, \cdots, g_\nu + m - 1,$$
$$g_\nu + m, g_\nu - m - 1, g_\nu - m - 2, \cdots, 1)$$

is equal to the density of (46).

Case II. $\nu = 2m$. Let

$$(48) \quad P_\nu: \quad S_1^{2m}, \cdots, S_n^{2m}$$

be such that

$$(49) \quad S'_{0_\nu+i} \equiv S_{2i+1} \quad (i = 0, 1, 2, \cdots, m - 1) \text{ and}$$
$$S'_{0_\nu-i} \equiv S_{2i} \quad (i = 1, \cdots, m).$$

Let $S_{k_\nu}^{2m} \equiv S_{2m+1}$. If $k_\nu \equiv g_\nu + m$, then let $P_{\nu+1} \equiv P_\nu$.

(1) Suppose $g_\nu + m + 1 \leq k_\nu \leq n$. Then by Lemma 1 or Lemma 2 the density of

$$(50) \quad (2m; 1, 2, \cdots, g_\nu + m - 1, k_\nu, k_\nu - 1, \cdots, g_\nu + m + 1,$$
$$g_\nu + m, k_\nu + 1, k_\nu + 2, \cdots, n)$$

is greater than or equal to that of P_ν since $r_{0_\nu+m-1}^{2m} \leq r_{k_\nu+1}^{2m}$ and $r_{k_\nu}^{2m} \leq r_{0_\nu+m}^{2m}$.

(2) Suppose $1 \leq k_\nu \leq g_\nu - m - 2$. Then by Lemma 1 or Lemma 2 the density of

$$(51) \quad (2m; 1, 2, \cdots, k_\nu - 1, g_\nu - m - 1, g_\nu - m - 2, \cdots, k_\nu + 1,$$
$$k_\nu, g_\nu - m, g_\nu - m + 1, \cdots, n)$$

is greater than or equal to that of P_ν; and the density of

$$
(52) \quad
\begin{aligned}
&(2m; 1, 2, \cdots, k_\nu - 1, g_\nu - m - 1, g_\nu - m - 2, \cdots, k_\nu + 1, k_\nu, \\
&\qquad g_\nu + m - 1, g_\nu + m - 2, \cdots, g_\nu - m, g_\nu + m, g_\nu + m + 1, \cdots, n)
\end{aligned}
$$

is greater than or equal to that of (51). Finally, the density of the reverse of (52)

$$
(53) \quad
\begin{aligned}
&(2m; n, n - 1, \cdots, g_\nu + m + 1, g_\nu + m, g_\nu - m, g_\nu - m + 1, \cdots, \\
&\qquad g_\nu + m - 2, g_\nu + m - 1, k_\nu, k_\nu + 1, \cdots, g_\nu - m - 2, \\
&\qquad\qquad g_\nu - m - 1, k_\nu - 1, k_\nu - 2, \cdots, 1)
\end{aligned}
$$

is equal to the density of (52).

Thus, depending on ν and k_ν, $P_{\nu+1}$ is taken as (44), (47), (50) or (53). P_2 is obtained from P_1, P_3 from P_2, $\cdots$, P_n from P_{n-1}. Thus we obtain an ordered set (27) satisfying the properties (a), (b) and (c) stated at the beginning of §5.

This completes the proof of part (1) of the theorem. It is clear from the statements of Lemmas 1 and 2 that the density of the packing (3) is greater than the density of any other packing (besides the reverse of (3)), if no two r's of (1.1) are equal. d_n is actually greater than d_1 in (27), if the strong inequalities hold at least once in the repeated application of Lemmas 1 and 2.

6. **Proof of part (2) of the theorem.** We shall prove part (2) of the theorem by constructing an ordered set of packings

$$
(54) \qquad Q_0, \cdots, Q_n
$$

such that (a) Q_i is obtained from Q_{i-1} by a rearrangement of the spheres of Q_{i-1} $(i = 1, \cdots, n)$, (b) Q_n is the packing (4), and (c) if d_i is the density of Q_i $(i = 0, 1, \cdots, n)$, then

$$
d_0 \geqq d_1 \geqq d_2 \geqq \cdots \geqq d_n.
$$

The packing Q_ν will here also be denoted by (28); and (28.1) will denote (28.2). Let Q_0 be a packing $S_{i_1}, \cdots, S_{i_n}$ of the spheres (1). Writing S_j^0 for S_{i_j} $(j = 1, \cdots, n)$, Q_0 becomes $S_1^0, \cdots, S_n^0$.

(1) If $S_1^0 \equiv S_1$, then let $Q_1 \equiv Q_0$. If $S_n^0 \equiv S_1$ then we can take Q_1 identical to the reverse of Q_0. Let $S_{g_0}^0 \equiv S_1$. Suppose $1 < g_0 < n$. Then the density of

$$
(55) \qquad (0; g_0, g_0 - 1, \cdots, 1, g_0 + 1, g_0 + 2, \cdots, n)
$$

is less than or equal to that of Q_0. For by Lemma 2

$$
(56) \qquad r_{g_0}^0 + 2(r(r_1^0 + r_{g_0+1}^0 - r))^{1/2} \geqq r_1^0 + 2(r(r_{g_0}^0 + r_{g_0+1}^0 - r))^{1/2}
$$

since $r_{g_0}^0 \leqq r_1^0$.

(2) Let Q_1: $S_1^1, \cdots, S_n^1$, be the packing (55). If $S_n^1 \equiv S_2$, let $Q_2 \equiv Q_1$. Let $S_{g_1}^1 \equiv S_2$. Suppose $1 < g_1 < n$. Then by Lemma 2, the density of

(57) $$(1; 1, 2, \cdots, g_1 - 1, n, n - 1, \cdots, g_1)$$

is less than or equal to that of Q_1 since $r_{g_1}^1 \leqq r_n^1$. Let $Q_2: S_1^2, \cdots, S_n^2$ be (57).

We now consider the general case, that is, we show how to obtain $Q_{\nu+1}$ from $Q_\nu: S_1^\nu, \cdots, S_n^\nu$ ($\nu \geqq 3$). Let

$$f(i) \equiv i \sin^2 (i\pi/2) + (n - i + 2) \cos^2 (i\pi/2)$$

and

$$g(i) \equiv (i + 1) \sin^2 (i\pi/2) + (n - i + 1) \cos^2 (i\pi/2).$$

Case I. $\nu = 2m$.
(1) $m = 2m_1$. Let

(58) $\quad S_i^\nu \equiv S_{f(i)}$ $(i = 1, \cdots, m)$ and $S_{n-i+1}^\nu \equiv S_{g(i)}$ $(i = 1, \cdots, m)$.

Let $S_{k_\nu}^{4m_1} \equiv S_{f(m+1)}$. If $k_\nu = m+1$ then let $Q_{\nu+1} \equiv Q_\nu$. Suppose $m+2 \leqq k_\nu \leqq n-m$. Then by Lemma 1, the density of

(59) $\quad (4m_1; 1, 2, \cdots, m, k_\nu, k_\nu - 1, \cdots, m + 1, k_\nu + 1, k_\nu + 2, \cdots, n)$

is less than or equal to that of Q_{4m_1} since $r_m^{4m_1} \geqq r_{k_\nu+1}^{4m_1}$ and $r_{m+1}^{4m_1} \geqq r_{k_\nu}^{4m_1}$.

(2) $m = 2m_1+1$. Let

$$S_i^\nu \equiv S_{f(i)} \ (i = 1, \cdots, m) \quad \text{and} \quad S_{n-i+1}^\nu \equiv S_{g(i)} \ (i = 1, \cdots, m).$$

Let $S_{k_\nu}^{4m_1+2} \equiv S_{f(m+1)}$. If $k_\nu = m+1$ then let $Q_{\nu+1} \equiv Q_\nu$. Suppose $m+2 \leqq k_\nu \leqq n-m$. Then by Lemma 1, the density of

(60) $\quad (4m_1 + 2; 1, 2, \cdots, m, k_\nu, k_\nu - 1, \cdots, m + 1, k_\nu + 1, k_\nu + 2, \cdots, n)$

is less than or equal to that of Q_{4m_1+2} since $r_m^{4m_1+2} \leqq r_{k_\nu+1}^{4m_1+2}$ and $r_{m+1}^{4m_1+2} \leqq r_{k_\nu}^{4m_1+2}$.

Case II. ν odd.
(1) $\nu = 2m+1$, $m = 2m_1$. Let

(61) $\quad S_i^\nu \equiv S_{f(i)}$ $(i = 1, \cdots, m + 1)$ and $S_{n-i+1}^\nu \equiv S_{g(i)}$ $(i = 1, \cdots, m)$.

Let $S_{k_\nu}^{2m+1} \equiv S_{g(m+1)}$. If $k_\nu = n-m$, let $Q_{\nu+1} \equiv Q_\nu$. Suppose $m+2 \leqq k_\nu < n-m$. Then by Lemma 1, the density of

(62) $\quad (2m + 1; 1, 2, \cdots, k_\nu - 1, n - m, n - m - 1, \cdots,$
$$k_\nu, n - m + 1, n - m + 2, \cdots, n)$$

is less than or equal to that of Q_{4m_1+1} since $r_{k_\nu-1}^{4m_1+1} \leqq r_{n-m+1}^{4m_1+1}$ and $r_{k_\nu}^{4m_1+1} \leqq r_{n-m}^{4m_1+1}$.

(2) $\nu = 2m-1$, $m = 2m_1$. Let

$$S_i^\nu \equiv S_{f(i)} \ (i = 1, \cdots, m) \quad \text{and} \quad S_{n-i+1}^\nu \equiv S_{g(i)} \ (i = 1, \cdots, m - 1).$$

Let $S_{k_\nu}^{2m-1} \equiv S_{g(m)}$. If $k_\nu = n-m+1$, let $Q_{\nu+1} \equiv Q_\nu$. Suppose $m+1 \leqq k_\nu \leqq n-m$. Then by Lemma 1, the density of

$$(63) \quad (4m_1 - 1; 1, 2, \cdots, k_\nu - 1, n - m + 1, n - m, \cdots,$$
$$k_\nu, n - m + 2, n - m + 3, \cdots, n)$$

is less than or equal to that of Q_{4m_1-1} since $r_{k_\nu-1}^{4m_1-1} \geqq r_{n-m+2}^{4m_1-1}$ and $r_{k_\nu}^{4m_1-1} \geqq r_{n-m+1}^{4m_1-1}$.

Thus $Q_{\nu+1}$ is taken as (59), (60), (62) or (63) according as ν is of the form $4m_1$, $4m_1+2$, $4m_1+1$, or $4m_1-1$. Q_3 is obtained from Q_2, Q_4 from Q_3, $\cdots$, Q_n from Q_{n-1}. Thus we obtain an ordered set (54), satisfying the properties stated at the beginning of §6.

This completes the proof of part (2) of the theorem. It is clear from the statements of Lemmas 1 and 2 that the density of (4) is less than the density of any other (incompressible) packing (besides the reverse of (4)), if no two r's of (1.1) are equal. d_n is actually less than d_1 in any set (54) if the strong inequalities hold at least once in the repeated application of Lemmas 1 and 2.

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